Institute of Actuaries of India

Subject SP6 – Financial Derivatives Principles

November 2023 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:

i) Differentiate the Put – Call parity equation and demonstrate the equivalence. [2]

ii) "C" carries more rights than "A" since in "C" we can exercise two options separately. In terms of price, they will be the same since the two pieces will have the same optimal exercise time. Hence "C" is preferred to "A". [2]

iii) Vega = s^2 * σ * T * gamma. If there is no volatility skew, then vega neutral implies that Gamma is zero. [2]

iv) Call increases in value while the put decreases in value. [2]

v) The price remains the same. [2]

vi) The asset is drift less so when the American option pays off there is precisely a 50% chance (in the risk neutral measure) that the European option will pay off too. So, the European is worth half as much as the American. [2]

[12 Marks]

Solution 2:

i) If the interest rates are zero, time dependence of volatility is irrelevant. Otherwise, the time dependence matters since the stock will drift in risk neutral measure and whether the volatility occurs before or after the drifting will make a difference. [3]

ii) Forward rates are always drift less – so we are back in the zero – interest case. [2]

[5 Marks]

Solution 3:

i) Default can be considered as a Bernoulli process as it either happens or doesn't happen. So the probability of default of A is P(A) means default can be considered as a Bernoulli distribution with expected value P(A) and standard deviation (P(A)*(1-P(A))^$^{0.5}$. Same applies to B. Since:

$$\rho_{AB} = \frac{E[X_A X_B] - E[X_A].E[X_B]}{\sigma_A \sigma_B}$$

$$E[X_A X_B]\ (which\ is\ the\ joint\ probability\ of\ default\ P_{AB}) = \rho_{AB} * \sqrt{P_A(1-P_A)P_B(1-P_B)P_B} + P_A P_B$$

[6]

ii) Probability of Company X defaulting is 3.6%
Probability of S Bank defaulting is 0.5%
Joint probability of default of X and S bank is 0.25*(0.036*(1-0.036)*.005*(1-0.005))0.5+0.005*0.036 = 0.346% which is the probability that K bank will incur a loss. [2]

iii) The expected loss, given a $100 million exposure with a probability of 0.346% is $100 million x 0.346% = $346,000 [1]

iv) If the default probabilities were independent, the expected loss would have been 0.5%*3.6%*100 Million = $ 18,000. The default correlation has increased the expected loss by ~19 times. [1]

[10 Marks]

Solution 4:

i) S_t follows Geometric Brownian Motion. This implies

$$S_t = S_0 \exp(\mu t + \sigma W_t).$$

Let us define $L_t = \ln(Z_t)$. Then

$$L_t = ln(B_t^{-1} S_t) = ln(B_t^{-1}) + ln(S_t) = ln(e^{-rt}) + ln(S_0 e^{\mu t + \sigma W_t})$$
$$= -rt + \mu t + \sigma W_t.$$

Hence,

$$dL_t = (\mu - r)dt + \sigma dW_t.$$

Now as $Z_t = f(L_t) = \exp(L_t)$, one can apply Ito's formula to get:

$$dZ_t = \left[\left((\mu - r)f'(L_t) + \frac{1}{2}\sigma^2 f''(L_t)\right)dt + \left(\sigma f'(L_t)\right)dW_t\right]$$

Substituting $f'(L_t) = f''(L_t) = \exp(L_t) = Z_t$, one gets

$$dZ_t = \left(\mu - r + \frac{\sigma^2}{2}\right)Z_t dt + \sigma Z_t dW_t.$$

[6]

ii) A portfolio is self-financing if and only if changes in its value depend only on changes in the prices of the assets constituting the portfolio.

Mathematically: If V_t denotes the value of the portfolio (ϕ_t, ψ_t), then the portfolio is self-financing if and only if

$$dV_t = \phi_t dS_t + \psi_t dB_t$$

A replicating strategy for X is a strategy which involves investing in specifiable quantities (ϕ_t, ψ_t) of stock and risk free bonds, such that the portfolio of (ϕ_t, ψ_t) of stocks and bonds will be self-financing the portfolio (ϕ_t, ψ_t) and will have terminal value equal to the magnitude of the claim; i.e. $V_T = \phi_T S_T + \psi_T B_T = X$.

This means that the portfolio's cash flows at the claim exercise date match the cash flows under the claim.

When the underlying stock follows a continuous geometric Brownian motion process, there is an additional technical constraint for the strategy to work; namely:

$$\int_0^T \phi_t^2 \sigma^2 dt < \infty.$$

[6]

iii) We have from (i)

$$Z_t = \left(\mu - r + \frac{\sigma^2}{2}\right)Z_t dt + \sigma Z_t dW_t.$$

The drift of this process is

$$\left(\mu - r + \frac{\sigma^2}{2}\right) \geq 0.$$

Since, the process is not driftless, it will not be a Martingale.

We can thus use the Cameron – Martin – Girsanov (CMG) theorem to convert the process into a Martingale. The CMG theorem states that there exists a probability measure Q, equivalent to the measure P (defined by the probability distribution of W_t), such that Z is a Martingale.

Thus, to apply the theorem we set

$$\gamma_t = \left(\mu - r + \frac{\sigma^2}{2}\right)$$

and then verify that γ_t is a pre-visible process and that

$$E_P\left[exp\left(\frac{1}{2}\int_0^T \gamma_t^2 dt\right)\right] < \infty.$$

Using the CMG theorem, the measure *Q*, that is equivalent to *P* is such that

$$\frac{dQ}{dP} = exp\left(-\frac{1}{2}\int_0^T \gamma_t^2 dt - \int_0^T \gamma_t dW_t\right)$$

and the Brownian motion

$$\widetilde{W_t} = W_t + \int_0^t \gamma_s ds$$

is a *Q*-measure Brownian motion.

Alternately, this can also be written as

$$d\widetilde{W_t} = \gamma dt + dW_t.$$

If one substitutes dW_t into the SDE for Z, one would get,

$$dZ_t = \sigma Z_t d\widetilde{W_{t.}}$$

This is the SDE for a driftless process under measure *Q*. Hence, Z is a Martingale under measure *Q*.

The next step in the construction of the replication strategy is to form the discounted expected claim process

$$E_t = E_Q(B_t^{-1}X|\boldsymbol{F}_t)$$

and show that this is a *Q*-measure Martingale as well. This can be done in a manner similar to the way Z_t is shown to be a *Q*-Martingale.

As both Z_t and E_t are *Q*-Martingales, the Martingale Representation Theorem (MRT) gives us a pre-visible process ϕ_t such that $dE_t = \phi_t \, dZ_t$.

It may be noted that in order to to apply the MRT we need to show that both Z_t and E_t are *Q*-Martingales and that the volatility of Z_t is non zero with probability 1.

The replication strategy then consists of holding ϕ_t units of stock and ψ_t risk free bonds, where ψ_t is given by

$$\psi_t = E_t - \phi_t Z_t$$

[6]

iv) The portfolio replicates the claim because the portfolio is self – financing and, at time *T*, when the claim falls due, the portfolios proceeds are

$$\phi_T S_T + \psi_T B_T = \phi_T S_T + (E_T - \phi_T Z_T)B_T = \phi_T S_T + (E_T - \phi_T B_T^{-1} S_T)B_T = E_T B_T$$
$$= E[B_T^{-1}X|F_T]B_T = X$$

i.e. the portfolio s proceeds will match the claim amount.

The value of the portfolio at any time *t* can be written

$$V_t = \phi_t S_{Tt} + \psi_t B_t = B_t E_{t.}$$

The SDE for V_t is given by

$$dV_t = d(B_t E_t) = B_t dE_t + E_t dB_t.$$

To show that the portfolio is self-financing, one needs to show that

$$dV_t = \phi_t S_t + \psi_t B_t$$

As the portfolio replicates the claim, the arbitrage-free condition requires that the value of the claim equals the value of the replicating strategy. Therefore, either of the above two SDEs gives the stochastic differential equation for the value of the claim.

[4]

[22 Marks]

Solution 5:

i) For each property, the notional amount of the property derivative contract should be set such that the change in the index value corresponds to the change in the property value.

Given the formula: Payoff = (Initial Index - Final Index) x Notional Amount

When the index drops by 1 point, the loss in property value should be equal to the payoff from the derivative. Therefore, the notional amount is calculated as: Notional Amount = Property Value / Initial Index

- **City centre building**: Notional Amount = ₹10 million / 1500 = ₹6,667 per point
- **Shopping mall**: Notional Amount = ₹15 million / 1200 = ₹12,500 per point
- **Office complex**: Notional Amount = ₹20 million / 2000 = ₹10,000 per point

[4]

ii) Using the formula for payoff and applying the correction factor: Adjusted Payoff = (Initial Index - Final Index) x Notional Amount x Correction Factor

- **City center building**: Adjusted Payoff = (1500 - 1450) x ₹6,667 x 1.05 = ₹350,000
- **Shopping mall**: Adjusted Payoff = (1200 - 1250) x ₹12,500 x 0.95 = ₹-593,750
- **Office complex**: Adjusted Payoff = (2000 - 1950) x ₹10,000 x 1.10 = ₹550,000

[4]

iii)

- **City centre** building: Change in Value = ₹10 million x 4% = -₹400,000
- **Shopping** mall: Change in Value = ₹15 million x 3% = ₹450,000
- **Office complex**: Change in Value = ₹20 million x 5% = -₹1 million

Interest on the loan: Interest = ₹5 million x 5% = ₹250,000

Net gain or loss: Net Gain/Loss = Total Adjusted Payoff + Change in Property Values - Interest Net Gain/Loss = (₹350,000 - ₹593,750 + ₹550,000) + (-₹400,000 + ₹450,000 - ₹1 million) - ₹250,000 Net Gain/Loss = ₹-943,750

With the correction factors applied, CityScape has a net marked to market loss of ₹943,750 after considering all factors.

[5]

[13 Marks]

Solution 6:

i) **Delta** Δ for a European call option:

$$\Delta = e^{qt} N(d_1)$$

$$d_1 = \frac{\ln\left(\frac{S}{k}\right) + \left(k - q + \frac{\sigma^2}{2}\right) \times t}{\sigma\sqrt{t}}$$

And Gamma

$$\Gamma = \frac{e^{-qt} n(d_1)}{S\sigma j\sqrt{t}}$$

$$\mathrm{n}(\mathrm{d}_1) = e^{\frac{-d_1^2}{2}} / \sqrt{2\pi}$$

S	10000
K	10500
t	0.5
r	5%
sigma	22%
dividend	2.50%
d1	-0.155500265
D2	-0.311063757
N(D1)	0.44
N(d2)	0.38
Delta	0.432769897
n(d1)	0.394148042
Gamma	0.000250221

Given the underlying stock price change is 3% of ₹ 10,000, which is:

Now, using the Taylor expansion to approximate the change in option price:

Change of option price = Δ×ΔS + ½ x Γ×(ΔS)2

Percent change	3%
S	10000
Change in price	300
Overall change	141

[3]

ii) Change in price due volatility can be estimated using vega.

$$V = Se^{-qt} n(d_1)\sqrt{T}$$

Volatility change	1.50%
Vega	27.52 per % change in implied volatility
Impact	27.52*1.5=41

[3]

[6 Marks]

Solution 7:

i) The LIBOR market model, also known as the Brace-Gatarek-Musiela (BGM) model, is a financial model used to model the evolution of forward interest rates. It's significant in pricing interest rate derivatives as it captures the dynamics of the entire forward rate curve, allowing for a more accurate and flexible pricing mechanism. [2]

ii) Multi-Currency LIBOR Market Model (LMM):

The multi-currency LIBOR market model is an extension of the single-currency LIBOR market model. It is used to model the evolution of a set of LIBOR rates for multiple currencies simultaneously. The significance of the multi-currency LMM lies in its ability to price and hedge interest rate derivatives that have exposure to multiple currencies. It is able to allow for the correlation between the entities. [2]

iii) Black's formula is given by

C=D(0,T)×L×[F(0,T) x N(d1)−K×N(d2)]

C is the caplet price.

D(0,*T*) is the discount factor from time 0 to time *T*.

L is the notional amount.

N(d) is the cumulative distribution function of the standard normal distribution.

K is the strike rate.

d_1 and d_2 are given by:

$$d_1 = \frac{\ln\left(\frac{F(0,T)}{K}\right) + \left(\frac{\sigma^2}{2}\right) \times T}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

For USD:

Given:

- *F(0,T) (forward LIBOR rate) = 3% or 0.03*
- *L*((notional amount) = 1 million
- *K* (strike rate) = 4% or 0.04
- *σ* (volatility) = 20% or 0.20
- T (time to maturity) = 1 year

F	3%
K	4%
sigma	20%
time	1
Discount	0.97
d1	-1.34
d2	-1.54
N(d1)	0.09
N(d2)	0.06
Notional	1000000
Value	225.60

For EUR

- *F*(0,*T*) (forward LIBOR rate) = 2% or 0.02
- r (discount rate) = 2% or 0.02
- *L*(0,*T*) (notional amount) = 1 million
- *K* (strike rate) = 3.5% or 0.035
- *σ* (volatility) = 18% or 0.18
- *T* (time to maturity) = 1 year

F	2%
K	3.50%

sigma	18%
time	1
Discount	0.980198673
d1	-3.01898
d2	-3.1989766
N(d1)	0.001268151
N(d2)	0.000689582
Notional	1000000
Value	1.20

[5]

iv) This would result increase in price of caplet. [1]

v) The multi-currency LMM can be extended to price cross-currency swaptions by incorporating the correlation between the interest rates of the two currencies and the exchange rate dynamics. Key considerations include:

- Correlation: The correlation between the interest rates of the two currencies plays a crucial role in determining the swaption's value.
- Quanto Adjustments: These are adjustments made to account for the risk of changes in the exchange rate between the two currencies.
- Model Calibration: The model needs to be calibrated to market data to ensure accurate pricing.
- Complexity: Pricing cross-currency swaptions is more complex than single-currency swaptions due to the additional factors involved.

[3]

[13 Marks]

Solution 8:

i) **Annual Premium Payment**:

Company BRONCO: Premium = CDS spread × Notional principal = 0.02 × ₹100 million = ₹2 million

Company LFA LEX: Premium = 0.025 × ₹50 million = ₹1.25 million [3]

ii) Payout in Case of Credit Event:

Company BRONCO:

Payout = (1 - Recovery rate) × Notional principal

= (1 - 0.40) × ₹100 million = ₹60 million

Company LFA LEX:

Payout = (1 - 0.40) × ₹50 million = ₹30 million [3]

iii) As company has already entered the contract widening of CDS spread would not have any impact on the payment and CDS payment would base on the initial CDS spread. We would have updated recovery which is given below.

	Spread	Notional	Payment (Million)
CDS Base spread LFA LEX	2.00%	100	2
CDS Base spread BRONCO	2.50%	50	1.25

Bond recovery rate

CDS Y1 spread LFA LEX	2.50%
CDS Y1spread BRONCO	1.50%
Recovery Rate Y1 LFA LEX	40%
Recovery Rate Y1 BRONCO	40%

Default payout	Recovery rate after change in CDS	Default loss	CDS payment (INR Million)
Bond 1	32%	68%	68
Bond 2	67%	33%	16.66667

	Year 1	Year 2	Recovery
Total payment	3.25	3.25	84.7
Total transaction	INR 78 m		

[5]

[11 Marks]

Solution 9:

i)

- Platykurtic distribution is a distribution with high kurtosis.
- Using a platykurtic distribution ensures that sufficient probability is allocated to the tails of the distribution in order to prevent underestimation of the effects of large price movements.
- Thus the approach can be quite useful when pricing out of the money options that become valuable when extreme price events take place in the underlying.

[3]

ii) Advantages

- By choosing an appropriate α value for the gamma distribution we can use a model that has a kurtosis higher than the corresponding lognormal model.
- A major drawback of Lognormal distribution is that their kurtosis is too low compared with observed movements. This drawback can be overcome by using the approach suggested by Prof Gull.
- The Gamma distribution spans all positive values which matches the range required for an asset price.

Disadvantages

- Gamma distribution is a less familiar model for practitioners to whom its properties may be unfamiliar. It may therefore be harder to establish the correct pricing formulae or to incorporate the model in software.
- Using the Gamma distribution can lead to more conservative price estimates which may not be in line with the generally accepted market practices, making it difficult to participate in the market.
- The Gamma model breaks the link with Geometric Brownian Motion, the underlying model from which lognormal distribution of asset prices is derived.

- The volatility parameter σ is a part of the SDE defining geometric Brownian motion, i.e. $dS_t=S_t(\mu dt+\sigma dB_t)$. There is no obvious corresponding SDE that generates the Gamma Model. So it will not be clear how to measure or interpret the volatility of the asset price using the gamma model.

[5]

[8 Marks]

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